On a process concerning inaccessible cardinals. I

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With the aid of the process of MAHLO one can get a hierarchy of the weakly inaccessible ordinal numbers. This hierarchy is based on the class of the regular ordinal numbers. MAHLO [1] defined a function $\pi_{\mu,\nu}$ such that the range of $\pi_{\mu,\nu}$ at $\nu=0$ is the class of the regular ordinal numbers, at $\nu=1$ is the class of the weakly inaccessible ordinal numbers and at $\nu=\eta$ is the class of the π_η-numbers, etc. We replace $\pi_{\mu,0}$ by a function $\varphi(\mu)$ the range of which is the class of the strongly (or weakly) inaccessible ordinal numbers and we define a process which is in the first two steps similar to the process of MAHLO.

We assume that each ordinal number is the set of all smaller ordinal numbers and that a cardinal number is an initial number ω_α. We denote by C the class of all the ordinal numbers. For every ordinal number α, $\omega_{\mathrm{cf}(\alpha)}$ denotes the least cardinal number β such that ω_α can be expressed as the sum of β cardinal numbers smaller than ω_α. Obviously $\mathrm{cf}(\alpha)\leqq\alpha$. If $\mathrm{cf}(\alpha)=\alpha$ then ω_α is said to be regular; otherwise they are singular. An ordinal number α is called a limit number if there is no β such that $\alpha=\beta+1$. We say that ω_α is a limit cardinal number if α is a limit number. A regular limit cardinal number is called weakly inaccessible. A weakly inaccessible ω_α is called strongly inaccessible if for every β, $\beta<\alpha$ implies $2^{\omega_\beta}<\omega_\alpha$. A subset M of ω_α (or a subclass M of C) is called stationary if $\omega_\alpha-M$ (or $C-M$) does not contain a closed subset (or subclass) confinal to ω_α (or C); otherwise it is called non-stationary.

If f is a function and x is an element of the domain of f then the value of f at x is denoted by $f(x)$. We consider such functions f for which $f(x)$ is an ordinal number. We denote by Rf the range of the function f and by Rf/α the set of the values of f which are smaller than α. We call a function f on $M\subset C$ into C regressive if for every $\gamma\in M$, $\gamma\neq0$, the inequality $f(\gamma)<\gamma$ holds, and $f(0)=0$ if $0\in M$. Let $A=\omega_\alpha$ or $A=C$, and let M be a subclass of A which is confinal to A. We call a function f on M into A strictly divergent if for every $\gamma\in A$ there is an ordinal number β such that $f(\xi)>\gamma$ for $\xi>\beta$.

In this paper we shall make reference to the following theorems.

Theorem I. *If $A=C$ or $A=\omega_\alpha$ with* $\mathrm{cf}(\alpha)>0$, *and M is a stationary subclass of A, then there is no strictly divergent function defined on M.* (See [2].)

Theorem II. *Let $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$, and let $\mathbf{K}$ be a class of non-empty and mutually disjoint non-stationary subclasses of A. If the class of the first elements of the classes belonging to $\mathbf{K}$ is non-stationary then the class $\bigcup\mathbf{K}$ is non-stationary.* (See [3].)

Theorem III. *If $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$, and $\{K_\gamma\}_{\gamma<\tau}$ $(\tau\in A)$ is a sequence of non-stationary subclasses of A then $\bigcup_{\gamma<\tau} K_\gamma$ is non-stationary.* (See [4].)

Clearly this theorem is a consequence of Theorem II.

Theorem IV. *Let $A=C$ or $A=\omega_\alpha$, where ω_α is a regular initial number with $\alpha>0$. Further let $\{K_\gamma\}_{\gamma\in A}$ be a sequence of non-empty and non-stationary subclasses of A such that the first elements x_γ $(\gamma\in A)$ of the classes K_γ $(\gamma\in A)$ are different and they are arranged in order of their magnitude i.e. $x_\gamma<x_\tau$ for $\gamma<\tau$. If the class $F=\{x_\gamma\}_{\gamma\in A}$ is non-stationary and if for every pair (γ,τ) of ordinal numbers with $\gamma<\tau$ the relation $x_\gamma\notin K_\tau$ holds, then $\bigcup_{\gamma\in A} K_\gamma$ is non-stationary.*

This is a consequence of Theorem II. Indeed, let

$$K'_\gamma=K_\gamma-\{x_\tau\}_{\tau>\gamma},$$

then the classes

$$K''_\gamma=K'_\gamma-\bigcup_{\delta<\gamma} K'_\delta$$

satisfy the conditions of Theorem II.

By the definition of the process we assume (as a sufficient condition) the following hypothesis:

(H) *The class of all the strongly (or weakly) inaccessible initial numbers is stationary.*

We assume that the strongly (or weakly) inaccessible initial numbers have been arranged in a strictly increasing sequence $\theta_0,\theta_1,\dots,\theta_\xi,\dots$. If we associate with every θ_ξ its index ξ, we obtain a strictly divergent function φ on the class of the strongly (or weakly) inaccessible numbers for which the inequality $\varphi(\gamma)\leqq\gamma$ holds. Thus it follows from the hypothesis (H) and Theorem I that the class of the fixed points of the function φ is stationary.

The process yields for the even ordinal numbers a sequence of matrices (a hyper-sequence) and for the odd ordinal numbers a matrix of matrices (a hyper-matrix).

The idea of our process can be loosely described as follows.

The 0-th step of the process is the strictly increasing sequence S_0 of the strongly (or weakly) inaccessible initial numbers.

The first step of the process is the matrix M_1 the rows of which will be defined recursively as follows. The 0-th row of M_1 is the sequence S_0. Let $\alpha>0$ denote a given ordinal number and suppose that we have already defined the ξ-th rows of M_1 for all $\xi<\alpha$. If $\alpha=\beta+1$ then the α-th row of M_1 is the strictly increasing sequence of all fixed points of the (strictly increasing) sequence of the elements of the β-th row of M_1. If α is a limit number then the α-th row of M_1 is the strictly increasing sequence of the elements of the intersection of all the ξ-th rows of M_1 with $\xi<\alpha$.

The class of the 0-th elements of the rows of M_1 is stationary. To show this, we define the matrix M'_1 with the aid of M_1 as follows. The α-th row of M is the sequence of the elements of the α-th row of M_1 which do not belong to the $(\alpha+1)$-th row of M_1. If we associate with every element of the α-th row of M'_1 its index corresponding to it in the α-th row of M_1 then we obtain a strictly divergent regressive function φ on the class of the elements of the α-th row of M_1. It follows from Theorem I that this class is non-stationary. The class of the rows of M_1' gives a decomposition of C into non-empty and mutually disjoint non-stationary subclasses of C. Therefore by Theorem II the class of the 0-th elements of the rows of M_1 is stationary.

The second step of the process is the hyper-sequence S_2 the elements of which we define recursively as follows. The 0-th element of S_2 is the matrix M_1. Let $\alpha > 0$ denote a given ordinal number and suppose that we have already defined the ξ-th matrix belonging to S_2 for all $\xi < \alpha$. We define the α-th matrix belonging to S_2 in the same way as we have defined the matrix M_1 starting from S_0 but, in the case $\alpha = \beta + 1$ we start instead of S_0 from the strictly increasing sequence of the 0-th elements of the rows of the β-th matrix belonging to S_2 and, in the case of a limit number α, from the strictly increasing sequence of the elements of the intersection of the classes of the 0-th elements of the rows of the ξ-th matrices belonging to S_2 with $\xi < \alpha$.

The class consisting of the elements $(0, 0)$ *of the matrices belonging to* S_2 *is stationary.* To show this, we define the hyper-sequence $S_2'' = \{N_0'', N_1'', \ldots\}$ with the aid of S_2 as follows. First we form the matrix N_α' starting with N_α in the same manner as we have formed M_1' starting with M_1. Thus, we obtain that the class of the elements of an arbitrary row of N_α' is non-stationary. After this we form the matrix N_α'' starting with N_α' in such a manner that we omit the rows of N_α' the 0-th elements of which belong to the 0-th row of $N_{\alpha+1}$. If we associate with the 0-th element of every row of N_α'' its index corresponding to it in the strictly increasing sequence of the 0-th elements of the rows of N_α then we obtain a strictly divergent regressive function on the class O_α of the 0-th elements of the rows of N_α''. Thus, by Theorem I the class O_α is non-stationary. Since no two different rows of N_α'' contain common elements and the class of the elements of an arbitrary row of N_α'' is non-stationary, it follows from Theorem II that the class of the elemets of the matrix N_α'' is non-stationary.

On the other hand, for every $0 < \beta < \alpha$ there corresponds to every element γ of N_α'' one, and only one row $R_\beta'(\gamma)$ of N_β' the 0-th elements of which is γ (and no element of N_α'' belongs to an N_β' with $\beta > \alpha$). Therefore, it follows from Theorem III that the union $\bigcup_{\beta<\alpha} R_\beta'(\gamma) = Q(\gamma)$ of the classes $R_\beta'(\gamma)$ $(\beta < \alpha)$ is non-stationary. In this way to every element γ of N_α'' there corresponds a non-stationary class $Q(\gamma)$ the smallest element of which is γ. Since the class of the elements of an arbitrary row of N_α'' is non-stationary, it follows from Theorem IV that the union Q of these non-stationary classes $Q(\gamma)$ corresponding to the elements γ of an arbitrary row of N_α'' is non-stationary. In this way to every row of N_α'' there corresponds a non-stationary class the smallest element of which is the 0-th element of the row. Since the class O_α of the 0-th elements of the rows of N_α'' is non-stationary, it follows from Theorem IV that the union U_α of the classes Q corresponding to the rows of N_α'' is non-stationary. In this manner to every matrix N_α'' there corresponds a non-stationary class U_α the smallest element of which is the element $(0, 0)$ of N_α''. Since the union of the classes U_α is equal to C, it follows from Theorem IV that the class of the $(0, 0)$-th elements of the matrices N_α, is stationary.

The third step of the process is the hyper-matrix $M_3 = (L_{\beta,\alpha})$ the rows of which we define recursively as follows. The 0-th row of M_3 is the hyper-sequence S_2. Let $\beta > 0$ denote a given ordinal number and suppose that we have already defined the ξ-th rows of M_3 for all $\xi < \beta$. We define the β-th row of M_3 in the same way as we have defined the hyper-sequence S_2 starting from S_0, but in the case $\beta = \gamma+1$ we start instead of S_0 from the strictly increasing sequence of the elements $(0, 0)$ of the matrices belonging to the γ-th row of M_3 and, in the case of limit numbers α, from the strictly increasing sequence of the elements of the intersection of the classes of the elements $(0, 0)$ of the matrices belonging to the ξ-th rows of M_3 with $\xi<\beta$.

The class formed by the elements $(0, 0)$ *of those matrices which are the* 0*-th elements of the rows of* M_3, *is stationary*. To show this, we define the hyper-matrices $M_3'=(L_{\beta,\alpha}')$, $M_3''=(L_{\beta,\alpha}'')$ and $M_3'''=(L'''_{\beta,\eta_\alpha^{(\beta)}})$ with the aid of M_3 as follows:

(1) We form the matrix $L_{\beta,\alpha}'$ starting with $L_{\beta,\alpha}$ in the same manner as we have fromed M_1' starting with M_1, thus we obtain that the class of the elements of an arbitrary row of $L_{\beta,\alpha}'$ is non-stationary.

(2) We form the matrix $L_{\beta,\alpha}''$ starting with $L_{\beta,\alpha}'$ in such a manner that we omit the rows of $L_{\beta,\alpha}'$ the 0-th elements of which belong to the 0-th row of $L_{\beta,\alpha+1}$; thus we obtain that the class of the 0-th elements of the rows of $L_{\beta,\alpha}''$ is non-stationary.

(3) We form the β-th row of M_3''' in such a manner that we keep (in the order of the elements of β-th row of M_3'') only those matrices $L_{\beta,\alpha}''$ the elements $(0, 0)$ of which belong to the 0-th row of the matrix $L_{\beta+1,0}$; thus we obtain that the class of the element $(0, 0)$ of the matrices belonging to the β-th row of M_3''' is non-stationary.

Consider now the hyper-sequence $S_\beta''=\{L_{\beta,0}'', L_{\beta,1}'', \dots\}$. For every $0<\gamma<\alpha$, there corresponds to every element η of $L_{\beta,\alpha}''$ one, and only one $R_\gamma'(\eta)$ of $L_{\beta,\gamma}'$ the 0-th element of which is η (and no element of $L_{\beta,\alpha}''$ belongs to an $L_{\beta,\gamma}'$ with $\gamma>\alpha$). Therefore, it follows from Theorem III that the union $\bigcup\limits_{\gamma<\alpha} R_\gamma'(\eta)=Q'(\eta)$ of the classes $R_\gamma'(\eta)$ $(\gamma<\alpha)$ is non-stationary. Similarly, there corresponds to every element η of $L_{\beta,\alpha}$ one, and only one row $R_{\tau,\gamma}''$ of $L_{\tau,\gamma}'$, where $\tau<\beta$ and $\gamma<\eta$, the 0-th element of which is η. Theorem III implies that $\bigcup\limits_{\tau<\beta}\ \bigcup\limits_{\gamma<\eta} R_{\tau,\gamma}''=Q''(\eta)$ is non-stationary. Put $Q(\eta)=Q'(\eta)\cup Q''(\eta)$. In this way to every element η of $L_{\beta,\alpha}''$ there corresponds a non-stationary class $Q(\eta)$ the smallest element of which is η. Since the class of the elements of an arbitrary row of $L_{\beta,\alpha}''$ is non-stationary, it follows from Theorem IV that the union Q of these non-stationary classes $Q(\eta)$ corresponding to the elements η of an arbitrary row of $L_{\beta,\alpha}''$ is non-stationary class the smallest element of which is the 0-th element of the row. Since the class of the 0-th elements of the rows of $L_{\beta,\alpha}''$ is non-stationary, it follows from Theorem IV that the union $U_{\beta,\alpha}$ of the classes Q corresponding to the rows of $L_{\beta,\alpha}''$ is non-stationary.

Since every element of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ belongs to the 0-th row of $L_{\beta,0}$, for every $\gamma<\beta$ there corresponds to every element μ of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ one, and only one element of S_γ'' the element $(0, 0)$ of which is μ (and no element of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ belongs to an element of S_γ with $\gamma>\beta$). Thus, for every $\gamma<\beta$, there corresponds to every element μ of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ one, and only one element of the sequence $U_\gamma=\{U_{\gamma,0}, U_{\gamma,1}, \dots\}$ the smallest element of which is μ. Since the class of the elements of an arbitrary row of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ is non-stationary, it follows from Theorem IV that the union of the non-stationary classes corresponding to the elements of an arbitrary row of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ is non-stationary. In this way to every row of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ there corresponds a non-stationary class the smallest element of which is the 0-th element of the row. Since the class of the 0-th elements of the rows of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ is non-stationary, it follows from Theorem IV that the union $P_{\beta,\eta_\alpha^{(\beta)}}$ of these non-stationary classes corresponding to the rows of $L'''_{\beta,\eta_\alpha^{(\beta)}}$ is non-stationary. In such a manner there are associated with $L'''_{\beta,\eta_\alpha^{(\beta)}}$ two non-stationary classes, $U_{\beta,\eta_\alpha^{(\beta)}}$ and $P_{\beta,\eta_\alpha^{(\beta)}}$. By Theorem III the union $R_{\beta,\eta_\alpha^{(\beta)}}$ of these classes is non-stationary. The smallest element of $R_{\beta,\eta_\alpha^{(\beta)}}$ is the element (0, 0) of $L'''_{\beta,\eta_\alpha^{(\beta)}}$. Since the class of the elements (0, 0) of the matrices belonging to the sequence $L'''_{\beta,\eta_0^{(\beta)}}, L'''_{\beta,\eta_1^{(\beta)}}, \ldots$ is non-stationary, it follows from Theorem IV that the union V_β of the classes $R_{\beta,\eta_0^{(\beta)}}, R_{\beta,\eta_1^{(\beta)}}, \ldots$ is non-stationary. The smallest element of V_β is the element (0, 0) of $L_{\beta,0}$. Since the union of the classes V_β is equal to C it follows from Theorem IV that the class formed by the elements (0, 0) of those matrices which are the 0-th elements of the rows of M, is stationary.

If we omit the rows of M_3''' in which the elements (0, 0) of the 0-th elements agree with their indices in the increasing sequence of the elements (0, 0) of the 0-th elements of the rows of M_3''' then we obtain the hyper-matrix $M_3^{(\mathrm{IV})}$. It can be proved that every strongly (or weakly) inaccessible initial number μ contained in the matrices belonging to $M_3^{(\mathrm{IV})}$ has the property that the set of the strongly (or weakly) inaccessible initial numbers smaller than μ is non-stationary in μ.

The process can be carried further; starting with the hyper-matrix M_3 we can form the hyper-sequence S_4 etc.

§ 1. The definition of the process

We assume that all the strongly (or weakly) inaccessible ordinal numbers are arranged in a strictly increasing sequence $\theta_0, \theta_1, \ldots, \theta_\mu, \ldots$ and we put $\varphi(\mu) = \theta_\mu$. We denote by $(\varphi(\mu))'$ the fixed points of the function $\varphi(\mu)$ i.e. all ordinal numbers μ, arranged according to their magnitude, for which $\varphi(\mu) = \mu$.

We define by transfinite induction the functions

$$f_0(\alpha^{(0)}),\ f_1(\alpha^{(0)}, \alpha^{(1)}),\ \ldots,\ f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)}),\ \ldots, \tag{1}$$

the variables η and $\alpha^{(\eta)}$ ranging over all ordinal numbers. Now, we define (1) as follows. We put $f(\alpha^{(0)}) = \varphi(\alpha^{(0)})$. We define the function $f_1(\alpha^{(0)}, \alpha^{(1)})$ by transfinite induction. Let

$$\begin{aligned} f_1(\alpha^{(0)}, 0) &= f_0(\alpha^{(0)}),\\ f_1(\alpha^{(0)}, \delta+1) &= (f_1(\alpha^{(0)}, \delta))',\\ Rf_1(\alpha^{(0)}, \lambda) &= \bigcap_{\varrho<\lambda} Rf_1(\alpha^{(0)}, \varrho) \text{ for any limit number } \lambda. \end{aligned}$$

The process by which we have constructed the function $f_1(\alpha^{(0)}, \alpha^{(1)})$ is called the first operation and we denote it by $\mathbf{\Gamma}_1$. Let now $\xi > 1$ be a given ordinal number and suppose that the operations $\mathbf{\Gamma}_\mu$ and the functions $f_\mu(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\zeta)}, \ldots, \alpha^{(\mu)})$, where $1 \leqq \mu < \xi$, have been already defined. We define the operation $\mathbf{\Gamma}_\xi$ and the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)})$ as follows. There are two possibilities:

a) ξ is an ordinal number of the first kind, i.e. $\xi = \tau + 1$;

b) ξ is an ordinal number of the second kind.

In the case a) let

$$f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}, 0) = f_\tau(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}).$$

Let now $\varkappa > 0$ be a given ordinal number and suppose that the functions $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}, \nu)$, where $0 \leqq \nu < \varkappa$, have already been defined. Then we define the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, 0, \varkappa)$ with the aid of the operation $\mathbf{\Gamma}_1$ as follows: Let

$$(2) \qquad f_{\tau+1}(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \varrho+1) = (f_{\tau+1}(0, \ldots, 0, \ldots, \alpha^{(\tau)}, \varrho))'$$

if $\varkappa = \varrho + 1$,

$$(3) \qquad Rf_{\tau+1}(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \varkappa) = \bigcap_{\nu<\varkappa} Rf_{\tau+1}(0, \ldots, 0, \ldots, \alpha^{(\tau)}, \nu)$$

if $\varkappa$ is a limit number;

$$f_{\tau+1}(\alpha^{(0)}, \eta+1, 0, \ldots, 0, \ldots, 0, \varkappa) = (f_{\tau+1}(\alpha^{(0)}, \eta, 0, \ldots, 0, \ldots, \varkappa))'$$

and

$$Rf_{\tau+1}(\alpha^{(0)}, \lambda, 0, \ldots, 0, \ldots, 0, \varkappa) = \bigcap_{\delta<\lambda} Rf_{\tau+1}(\alpha^{(0)}, \delta, 0, \ldots, 0, \ldots, 0, \varkappa)$$

if λ is a limit number. So we obtain the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, 0, \varkappa)$ and applying step by step the operations $\mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \ldots, \mathbf{\Gamma}_\xi, \ldots$ $(\mu < \xi)$ we obtain the functions

$$\begin{gathered} f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, 0, \ldots, 0, \ldots, 0, \varkappa) \\ \vdots \\ f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, 0, \ldots, 0, \ldots, 0, \varkappa) \\ \vdots \end{gathered}$$

i.e. the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}, \varkappa)$. Thus we have defined the function $f_{\tau+1}(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\tau)}, \varkappa)$ for all ordinal numbers $\varkappa$.

In the case b) we define the function

$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, 0)$$

as follows: Let

$$\begin{aligned} f_\xi(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0) &= f_0(\alpha^{(0)}) \\ f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, 0) &= f_1(\alpha^{(0)}, \alpha^{(1)}) \\ &\vdots \\ f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, 0, \ldots, 0, \ldots, 0) &= f_\mu(\alpha^{(0)} \alpha^{(1)}, \ldots, \alpha^{(\mu)}). \\ &\vdots \end{aligned}$$

Let now $\varkappa > 0$ be a given ordinal number and suppose that the functions $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \nu)$, where $0 \leqq \nu < \varkappa$, have already been defined. Then we define the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, \varkappa)$ with the aid of the operation $\mathbf{\Gamma}_1$ as follows: Let

$$(4) \qquad Rf_\xi(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varrho+1) = \bigcap_{\mu<\xi} Rf_\xi(0, \ldots, 0, \ldots, \alpha^{(\mu)}, 0, \ldots, 0, \ldots, \varrho)$$

if $\varkappa = \varrho + 1$,

$$(5) \qquad Rf_\xi(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varkappa) = \bigcap_{\nu<\varkappa} Rf_\xi(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu)$$

if $\varkappa$ is a limit number,

$$f_\xi(\alpha^{(0)}, \eta+1, 0, \ldots, 0, \ldots, \varkappa) = (f_\xi(\alpha^{(0)}, \eta, 0, \ldots, 0, \ldots, \varkappa))'$$

and

$$Rf_\xi(\alpha^{(0)}, \lambda, 0, \ldots, 0, \ldots, \varkappa) = \bigcap_{\delta<\lambda} Rf_\xi(\alpha^{(0)}, \delta, 0, \ldots, 0, \ldots, \varkappa)$$

if λ is a limit number. In this way we obtain the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, \varkappa)$ and applying step by step the operations $\mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \ldots, \mathbf{\Gamma}_\mu, \ldots$ $(\mu<\xi)$ we obtain the functions

$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, 0, \ldots, 0, \ldots, \varkappa)$$
$$\vdots$$
$$f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, 0, \ldots, 0, \ldots, \varkappa)$$
$$\vdots$$

i.e. the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \varkappa)$ for all ordinal numbers $\varkappa$. The process by which we have constructed the function $f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)})$ is called the ξ-th operation and we denote it by $\mathbf{\Gamma}_\xi$.

In this manner we have defined the functions

$$f_0(\alpha^{(0)}),\ f_1(\alpha^{(0)}, \alpha^{(1)}), \ldots, f_\xi(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\xi)}), \ldots$$

for the variables ξ, $\alpha^{(\xi)}$ ranging over all ordinal numbers. Consider now the functions $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\mu)}, \ldots, \alpha^{(\eta)})$. For $\mu+1\leqq\eta$ let us denote by $A_{\mu,\eta}=A_{\mu,\eta}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$ the set of ordinal numbers $\alpha_\xi^{(\mu)}=\alpha_\xi^{(\mu)}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$, arranged according to their magnitude (i.e. $\alpha_\xi^{(\mu)}<\alpha_{\xi+1}^{(\mu)}$), for which

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\xi^{(\mu)}, \alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})>\alpha_\xi^{(\mu)}$$

and for $\mu=\eta$ let us benote by $A_{\eta,\eta}(0)$ the set of the ordinal numbers $\alpha_\xi^{(\eta)}=\alpha_\xi^{(\eta)}(0)$, arranged according to their magnitude, for which

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\xi^{(\eta)})>\alpha_\xi^{(\eta)}.$$

Further, for $\mu+1\leqq\eta$ let us denote by $n_{\mu,\eta}=n_{\mu,\eta}(\alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)})$ the smallest ordinal number γ for which

$$\gamma=f_\eta(0, \ldots, 0, \ldots, \gamma, \alpha^{(\mu+1)}, \ldots, \alpha^{(\eta)}),$$

and for $\mu=\eta$ let us denote by $n_{\eta\eta}(0)$ the smallest ordinal number γ for which

$$\gamma=f_\eta(0, \ldots, 0, \ldots, \gamma).$$

Let $\mu\leqq\eta$ and let $\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}$ be given ordinal numbers. We denote by

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

the values of $f_\xi(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})<\alpha$ for which the conditions

$$\begin{aligned}
\alpha_\zeta^{(0)} &= \alpha_\xi^{(0)}(\alpha_\zeta^{(1)}, \alpha_\tau^{(2)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})\in A_{0,\eta}(\alpha_\zeta^{(1)}, \alpha_\tau^{(2)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}),\\
\alpha_\zeta^{(1)} &= \alpha_\zeta^{(1)}(\alpha_\tau^{(2)}, \alpha_\varrho^{(3)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \alpha^{(\eta)})\in A_{1,\eta}(\alpha_\tau^{(2)}, \alpha_\varrho^{(3)}, \ldots, \alpha_\varphi^{(\nu)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}),\\
&\ \vdots\\
\alpha_\varphi^{(\nu)} &= \alpha_\varphi^{(\nu)}(\alpha_\psi^{(\nu+1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})\in A_{\nu,\eta}(\alpha_\psi^{(\nu+1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})\\
&\ \vdots
\end{aligned}$$

hold.

§ 2. Results

We prove now the following

Theorem 1. *If $i<\omega$ and $\alpha=n_{i,i}(0)$ then the set of the ordinal numbers of the form $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})<\alpha$ is non-stationary in α.*

Proof. We distinguish three cases:

(a) $i=0$, (b) $i=1$, (c) $i\geqq 2$.

Case (a). By definition, $n_{0,0}(0)$ is the smallest ordinal number ϱ for which $\varrho=f_0(\varrho)$. Therefore, the set of the ordinal numbers of the form $f_0(\alpha^{(0)})<\alpha$ is equal to

$$M = \{f_0(\alpha_\xi^{(0)}) : \alpha_\xi^{(0)}\in A_{0,0}(0) \quad \text{and} \quad \alpha_\xi^{(0)}<\alpha\}. \tag{6}$$

We define a function g on M as follows:

$$g\big(f_0(\alpha_\xi^{(0)})\big) = \alpha_\xi^{(0)}.$$

It is easy to see that the function g is strictly divergent and regressive on the set (6). Thus, it follows from Theorem I that the set (6) is non-stationary in α.

Case (b). By definition, $n_{1,1}(0)$ is the smallest ordinal number ϱ for which $\varrho=f_1(0, \varrho)$. Therefore, the set of the ordinal numbers of the form $f_1(\alpha^{(0)}, \alpha^{(1)})<\alpha$ is equal to

$$\bigcup_{\beta<\alpha}\{f_1(\alpha_\xi^{(0)}(\beta), \beta) : \quad \alpha_\xi^{(0)}(\beta)\in A_{0,1}(\beta) \quad \text{and} \quad \alpha_\xi^{(0)}(\beta)<\alpha\}. \tag{7}$$

We must prove that this set is non-stationary in α.

First we prove that the set $M=\{f_1(0, \beta) : \beta<\alpha\}$ is non-stationary. For this reason we define a function g on M by writing $g(f_1(0, \beta))=\beta$. Since $f_1(0, \tau)$ is a strictly increasing function of the variable τ and for every ordinal number $\beta<\alpha$ the inequality $\beta<f_1(0, \beta)$ holds, it follows that g is a strictly divergent and regressive function on M. Therefore Theorem I implies that the set M is non-stationary in α.

Our next purpose is to show that, for each $\beta<\alpha$, the set

$$N(\beta) = \{f_1(\alpha_\xi^{(0)}(\beta), \beta) : \alpha_\xi^{(0)}(\beta)<\alpha\}$$

is non-stationary in α. For each $\beta<\alpha$ we define a function g_β on $N(\beta)$ as follows:

$$g_\beta\big(f_1(\alpha_\xi^{(0)}(\beta), \beta)\big) = \alpha_\xi^{(0)}(\beta).$$

Since for a given β the inequalities

$$\alpha_\xi^{(0)}(\beta) < \alpha_{\xi+1}^{(0)}(\beta)$$

and

$$\alpha_\xi^{(0)}(\beta) < f_1\big(\alpha_\xi^{(0)}(\beta), \beta\big) < f_1\big(\alpha_{\xi+1}^{(0)}(\beta), \beta\big)$$

hold, it follows that g_β is strictly divergent and regressive on $N(\beta)$. Therefore Theorem I implies that the set $N(\beta)$ $(\beta<\alpha)$ is non-stationary in α.

We are now ready to prove that the set (7) is non-stationary in α. Observe that the set of the first elements of the sets $N(\beta)$ with $\beta<\alpha$ is $\{f_1(0, \beta) : \beta<\alpha\}$. Thus, the preceding considerations imply that the set (7) is the union of non-empty and mutually disjoint non-stationary sets (namely the sets $N(\beta)$ with $\beta<\alpha$) the set of the first elements of which is non-stationary. Therefore Theorem II implies that the set (7) is non-stationary in α. Hence, in the case (b), the proof is complete.

Case (c). Let $i \geqq 2$ be a given natural number. Denote by $\gamma(\beta)$ the value $f_i(0, \ldots, 0, \ldots, \beta)$. We begin the proof by showing the validity of the following statement.

(i) *Assume that* $\beta \neq 0$. *Then* $\gamma(\beta)$ *satisfies the equality*

$$\gamma(\beta)=f_i(0, \ldots, 0, \gamma(\beta), \psi^{(j)}, \ldots, \psi^{(k)}, \ldots, \psi^{(i)})$$

for every j, $1 \leqq j \leqq i$, *provided that* $\psi^{(i)}<\beta$ *and* $\psi^{(k)}<\gamma(\beta)$ $(j \leqq k<i)$.

Since $\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \beta)$, we have

(8) $$\gamma(\beta) \in Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \beta).$$

It follows from (2) and (3) that

(9) $$\gamma \in Rf_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu)$$

for every $\nu<\beta$. First we show that $\gamma(\beta)$ satisfies the equality

(10) $$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \nu)$$

for every $\nu<\beta$. If not, then there are two ordinal numbers $\nu_0<\beta$ and $\tau_0<\gamma(\beta)$ such that

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \tau_0, \nu_0).$$

By (2)

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu_0+1)=(f_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu_0))'.$$

This means that

(11) $$\gamma(\beta) \notin Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu_0+1).$$

In virtue of the relation (8), we conclude that $\nu_0+1<\beta$. On the other hand it follows from (11) and the construction of $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})$ that

$$\gamma(\beta) \notin Rf_i(0, \ldots, 0, \ldots, \alpha^{(i-1)}, \nu_0+1).$$

But this contradicts the fact that the relation (9) holds for every $\nu<\beta$ and we conclude that $\gamma(\beta)$ satisfies (10) for every $\nu<\beta$.

Let now l be a natural number for which $0<l<i$. Assume that whenever $\psi^{(i)}<\beta$ and $\psi^{(m)}<\gamma(\beta)$ $(l+1 \leqq m<i)$, then

(12) $$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l+1)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)}).$$

Since $\gamma(\beta)=f_i(0, \ldots, 0, 0, \ldots, \gamma(\beta), \psi^{(i)})$ for every ordinal number $\psi^{(i)}<\beta$, it remains to prove that this assumption implies the equality

$$\gamma(\beta)=f_i(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)})$$

for $\psi^{(i)}<\beta$ and $\psi^{(m)}<\gamma(\beta)$, where $l \leqq m<i$.

It follows from the definition of $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})$ that, for given $\psi^{(l+1)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)}$, the equalities

$$(13) \qquad Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l+1)}, \ldots, \psi^{(i)}) = \\ = \bigcap_{\mu<\gamma(\beta)} Rf_i(0, \ldots, 0, \ldots, \alpha^{(l-1)}, \mu, \psi^{(l+1)}, \ldots, \psi^{(i)})$$

and

$$(14) \qquad f_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu+1, \psi^{(l+1)}, \ldots, \psi^{(i)}) = \\ = ((f_i(0, \ldots, 0, \ldots, \alpha^{(l+1)}, \mu, \psi^{(l+1)}, \ldots, \psi^{(i)}))'$$

hold. By (12) and (13) we obtain for given $\psi^{(m)}$ $(l+1 \leqq m \leqq i)$, where $\psi^{(m)} < \gamma(\beta)$ $(l+1 \leqq m < i)$ and $\psi^{(i)} < \beta$, and for every $\mu < \gamma(\beta)$ that

$$(15) \qquad \gamma(\beta) \in Rf_i(0, \ldots, 0, \ldots, \alpha^{(l-1)}, \mu, \psi^{(l+1)}, \ldots, \psi^{(i)}).$$

Now we show that for given $\psi^{(m)}$ $(l+1 \leqq m \leqq i)$, where $\psi^{(m)} < \gamma(\beta)$ $(l+1 \leqq m < i)$ and $\psi^{(i)} < \beta$, and for every $\mu < \gamma(\beta)$ the ordinal number $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta) = f_i(0, \ldots, 0, \ldots, \gamma(\beta), \mu, \psi^{(l+1)}, \ldots, \psi^{(i)}).$$

In the contrary case there are two ordinal numbers, $\mu_0 < \gamma(\beta)$ and $\tau_0 < \gamma(\beta)$, such that

$$\gamma(\beta) = f_i(0, \ldots, 0, \ldots, \tau_0, \mu_0, \psi^{(l+1)}, \ldots, \psi^{(i)}).$$

Hence, by (14), we have

$$\gamma(\beta) \notin Rf_i(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu_0+1, \psi^{(l+1)}, \ldots, \psi^{(i)}).$$

Consequently, by the definition of $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(i)})$,

$$\gamma(\beta) \notin Rf_i(0, \ldots, 0, \ldots, \alpha^{(l-1)}, \mu_0+1, \psi^{(l+1)}, \ldots, \psi^{(i)}).$$

Since $\gamma(\beta)$ is a limit number, and in virtue of this $\mu_0+1 < \gamma(\beta)$, the last relation contradicts the fact that (15) holds for every $\mu < \gamma(\beta)$. Thus, we may conclude that if $0 < l < i$, $\psi^{(i)} < \beta$ and $\psi^{(m)} < \gamma(\beta)$ for $l \leqq m < i$, then

$$\gamma(\beta) = f_i(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(l)}, \ldots, \psi^{(i)}).$$

The proof of the statement (i) is complete.

The same method can be used to prove the following statement.

(ii) *Assume that* $\alpha^{(k)}, \ldots, \alpha^{(i)}$ $(0 < k \leqq i)$ *are given ordinal numbers and* $\alpha^{(k)} \neq 0$. *Then* $\gamma = f_i(0, \ldots, 0, \alpha^{(k)}, \ldots, \alpha^{(i)})$ *satisfies the equality*

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k)}, \alpha^{(k+1)}, \ldots, \alpha^{(i)})$$

for every j $(1 \leqq j \leqq k)$, *provided that* $\psi^{(k)} < \alpha^{(k)}$ *and* $\psi^{(m)} < \gamma$ *for each* m $(j \leqq m \leqq k)$.

Now we proceed to prove the following statement.

(iii) *Assume that* $\underline{\alpha}^{(0)}, \ldots, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)}$ *are given ordinal numbers,* $\underline{\alpha}^{(0)} \neq 0$ *and* $\underline{\alpha}^{(k)} \neq 0$. *Then* $\gamma = f_i(\underline{\alpha}^{(0)}, 0, \ldots, 0, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})$ *satisfies the equality*

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$$

for every j $(1 \leqq j \leqq k)$, *provided that* $\psi^{(k)} < \underline{\alpha}^{(k)}$ *and* $\psi^{(m)} < \gamma$ *for each* m $(j \leqq m \leqq k)$.

Let us denote by λ the ordinal numbers $\underline{\alpha}^{(k)}$. It follows from the definition $f_i(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(k)})$ that

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \varrho+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \varrho, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\lambda = \varrho+1$, and

$$Rf_i(\alpha^{(0)}, 0, \ldots, 0, \lambda, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \bigcap_{\nu<\lambda} Rf_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}).$$

These imply that for every $\nu<\lambda$

$$\gamma \in Rf_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}). \tag{16}$$

Since

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \nu+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\nu<\lambda$ and

$$f_i(\alpha^{(0)}, 0, \ldots, 0, \nu+2, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) = \big(f_i(0, \ldots, 0, \alpha^{(k-1)}, \nu+1, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big)'$$

for $\nu+1<\lambda$, the relation (16) implies that, for every $\nu<\lambda$,

$$\gamma = f_i(0, \ldots, 0, \gamma, \nu, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}).$$

Thus, by (ii), we obtain that for every $0<j\leqq k$

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(i)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}),$$

provided that $\psi^{(k)}<\underline{\alpha}^{(k)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k)$.

Now we can prove the following statement.

(iv) *Let $\{k_l\}_{l\leqq s}$ $(s\leqq i)$ be the strictly increasing sequence of the natural numbers $k\leqq i$ for which $\alpha^{(k)}\neq 0$. Assume that $k_0=0$. Then $\gamma=f_i(\alpha^{(0)}, \ldots, \alpha^{(i)})$ satisfies the equality*

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}, 0, \ldots, 0, \alpha^{(k_{l+1})}, \ldots, \alpha^{(i)})$$

for every l $(1\leqq l\leqq s)$ and for every j $(1\leqq j\leqq k_l)$, provided that $\psi^{(k_l)}<\alpha^{(k_l)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k_l)$.

Indeed, if (iv) is true for a fixed l, $1\leqq l<s$, then

$$\gamma = f_i(\gamma, 0, \ldots, 0, \alpha^{(k_l)}, 0, \ldots, 0, \alpha^{(k_l+1)}, \ldots, \alpha^{(i)}).$$

If we apply (iii) for $\underline{\alpha}^{(0)}=\gamma$ then we obtain that the number satisfies the equality

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l+1)}, 0, \ldots, 0, \alpha^{(k_l+2)}, \ldots, \alpha^{(i)})$$

for every j $(1\leqq j\leqq k_{l+1})$, provided that $\psi^{(k_l+1)}<\alpha^{(k_l+1)}$ and $\psi^{(m)}<\gamma$ for each m $(j\leqq m<k_{l+1})$. This proves the statement (iv).

Now we proceed the proof of Theorem 1 by showing that the set

$$Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha \tag{17}$$

is non-stationary in α. We define a function on $M=Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha$ by writing

$$g\big(f_i(0, \ldots, 0, \ldots, \beta)\big)=\beta.$$

Since $f_i(0, \ldots, 0, \ldots, \tau)$ is a strictly increasing function of the variable τ and for every $\beta<\alpha$ the inequality

$$\beta<f_i(0, \ldots, 0, \ldots, \beta)$$

holds, we obtain that the function g is strictly divergent and regressive on M. Therefore Theorem I implies that the set (17) is non-stationary in α.

After this we prove the following statement.

(v) *For every k, $0<k\leqq i$, the set*

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α, where $\underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)}$ are given ordinal numbers $<\alpha$.

By definition

$$A_{k,i}(\alpha^{(k+1)}, \ldots, \alpha^{(i)}) = \{\alpha_\varrho^{(k)}:\ \ f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \alpha^{(k+1)}, \ldots, \alpha^{(i)}) > \underline{\alpha}_\varrho^{(k)}\},$$

where $\alpha_\varrho^{(k)} = \alpha_\varrho^{(k)}(\alpha^{(k+1)}, \ldots, \alpha^{(i)})$ is a strictly increasing function of ϱ for given $\alpha^{(k+1)}, \ldots, \alpha^{(i)}$.

Consider now the sets

$$B_k = Rf_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(l)}, \ldots, \underline{\alpha}^{(i)})/\alpha =$$
$$= \{f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})<\alpha:\ \ \alpha_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\},$$

where $0\leqq k<i$ and $\underline{\alpha}^{(k)}$ $(k+1\leqq l\leqq i)$ are given ordinal numbers $<\alpha$. We define the functions g_k $(k=0, 1, \ldots, i-1)$ on the sets B_k $(k=0, 1, \ldots, i-1)$ as follows:

$$g_k\big(f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})\big) = \alpha_\varrho^{(k)}.$$

It follows from the definition of the $A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ $(k\leqq i-1)$ that the inequalities

$$\alpha_\varrho^{(k)}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) < \alpha_{\varrho+1}^{(k)}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$$

and

$$\alpha_\varrho^{(k)} < f_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)}) < f_i(0, \ldots, 0, \alpha_{\varrho+1}^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$$

hold for $k=0, 1, \ldots, i-1$. These imply that the functions g_k $(k=0, 1, \ldots, i-1)$ are strictly divergent and regressive on the sets B_k $(k=0, 1, \ldots, i-1)$. Therefore Theorem I implies that the sets B_k $(k=0, 1, \ldots, i-1)$ are non-stationary in α.

The preceding considerations show that, by given $\underline{\alpha}^{(1)}, \ldots, \alpha^{(k)}, \ldots, \underline{\alpha}^{(i)}<\alpha$, the sets

$$\begin{aligned} B_0 &= Rf_i(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha \\ &\vdots \\ B_k &= Rf_i(0, \ldots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha \\ &\vdots \end{aligned}$$

are non-stationary in α.

Suppose now that set

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}^{(k)}, \ldots, \underline{\alpha}^{(i)})/\alpha, \ \text{where}\ 0<k<i,$$

is non-stationary in α. This means that for every fixed $\underline{\alpha}_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ the set

$$Rf_i(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\tau^{(k-1)}, \underline{\alpha}_\varrho^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α. On the other hand it is easy to verify that for any two different elements $\underline{\alpha}_{\varrho}^{(k)}$ and $\underline{\alpha}_{\sigma}^{(k)}$ the sets

$$Rf_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\tau}^{(k-1)}, \underline{\alpha}_{\varrho}^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

and

$$Rf_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\tau}^{(k-1)}, \underline{\alpha}_{\sigma}^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

have no common elements.

Since the set of the first elements of the sets

$$Rf_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\tau}^{(k-1)}, \alpha_{\varrho}^{(k)}, \underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

with $\alpha_{\varrho}^{(k)} \in A_{k,i}(\underline{\alpha}^{(k+1)}, \ldots, \underline{\alpha}^{(i)})$ is equal to B_k we obtain from Theorem III that the union of these sets is non-stationary in α.

Thus we have proved the statement (v).

Since the set $Rf_i(0, \ldots, 0, \ldots, \beta)/\alpha$ is non-stationary in α, we obtain from (v) that the set

$$M = Rf_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\sigma}^{(i)})/\alpha$$

is non-stationary in α.

Consider now an arbitrary element $\gamma = f_i(\underline{\alpha}^{(0)}, \ldots, \underline{\alpha}^{(i)}) \neq 0$ of M. Let $\{k_l\}_{l \leqq s}$ $(s \leqq i)$ be the strictly increasing sequence of the natural numbers k, $0 \leqq k \leqq i$, for which $\underline{\alpha}^{(k)} \neq 0$. Let us denote by n_0 the smallest natural number $l \leqq s$ for which $k_l \geqq 2$. Then the statements (i)—(iv) imply that γ satisfies the equality

$$\gamma = f_i(0, \ldots, 0, \gamma, \psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}, \underline{\alpha}^{(k_l+1)}, \ldots, \underline{\alpha}^{(i)}) \tag{18}$$

for every l $(n_0 \leqq l \leqq s)$ and for every j $(1 \leqq j \leqq k_l)$, provided that $\psi^{(k_l)} < \underline{\alpha}^{(k_l)}$ and $\psi^{(m)} < \gamma$ for each m $(j \leqq m < k_l)$.

Let us denote by $S_{l,j}$, where $n_0 \leqq l \leqq s$ and $1 \leqq j \leqq k_l$, the set of the sequences

$$(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$$

such that $\psi^{(k_l)} < \underline{\alpha}^{(k_l)}$ and $\psi^{(m)} < \gamma$ for $j \leqq m \leqq k_l$. It is clear that the power of the set $S_{l,j}$ is $\leqq \gamma^i = \gamma$.

It follows from the statement (v) that, for any element $(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$ of $S_{l,j}$, the set

$$B(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)}) = Rf_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\tau}^{(j-1)}, \gamma, \psi^{(j)}, \ldots, \psi^{(k_l)}, \underline{\alpha}^{(k_l+1)}, \ldots, \underline{\alpha}^{(i)})/\alpha$$

is non-stationary in α.

Since $\gamma < \alpha$, Theorem II implies that the set

$$B(\gamma) = \bigcup_{l \leqq s} \bigcup_{j \leqq k_l} \bigcup_{\psi^{(j)} < \gamma} \cdots \bigcup_{\psi^{(m)} < \gamma} \cdots \bigcup_{\psi^{(k_l)} < \underline{\alpha}^{(k_l)}} B(\psi^{(j)}, \ldots, \psi^{(m)}, \ldots, \psi^{(k_l)})$$

is non-stationary in α. On the other hand, by (18), the smallest element of the set $B(\gamma)$ is γ.

In this manner we have associated with every element $\gamma = f_i(\alpha_{\xi}^{(0)}, \alpha_{\zeta}^{(1)}, \ldots, \alpha_{\sigma}^{(i)})$ of M a set $B(\alpha_{\xi}^{(0)}, \alpha_{\xi}^{(i)}, \ldots, \alpha_{\sigma}^{(i)})$ the smallest element of which is γ.

It only remains to prove that

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. Since the set M is non-stationary in α, the sets

$$\begin{aligned} B_0 &= Rf_i(\alpha_\xi^{(0)}, \underline{\alpha}_\zeta^{(1)}, \dots, \underline{\alpha}_\sigma^{(i)})/\alpha \\ &\vdots \\ B_k &= Rf_i(0, \dots, 0, \alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})/\alpha \\ &\vdots \end{aligned}$$

are non-stationary in α, where $\underline{\alpha}_\zeta^{(1)}, \dots, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)}$ are fixed ordinal numbers $<\alpha$. Since B_1 is non-stationary in α and the set of the smallest element of the sets $B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \underline{\alpha}_\varphi^{(2)}, \dots, \underline{\alpha}_\xi^{(i)}) = B(\alpha_\sigma^{(i)})$ is B_1, Theorem IV implies that the set

$$\bigcup_{\alpha_\xi^{(0)}<\alpha} \bigcup_{\alpha_\zeta^{(1)}<\alpha} B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \underline{\alpha}_\varphi^{(2)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α.

Suppose now that the set

$$B(\underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)}) = \bigcup_{\alpha_\xi^{(0)}<\alpha} \cdots \bigcup_{\alpha_\varrho^{(k-1)}<\alpha} B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \dots, \alpha_\varphi^{(k-1)}, \underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α. It is easy to verify that the smallest element of the set $B(\underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$ is $f_i(0, \dots, 0, \underline{\alpha}_\varrho^{(k)}, \dots, \underline{\alpha}_\sigma^{(i)})$. Thus the set of the first elements of the sets $B(\alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$ with $\alpha_\varrho^{(k)} \in A_{k,i}(\underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$ is equal to B_k. Therefore Theorem IV implies that the set

$$\bigcup_{\alpha_\varrho^{(k)}<\alpha} B(\alpha_\varrho^{(k)}, \underline{\alpha}_\tau^{(k+1)}, \dots, \underline{\alpha}_\sigma^{(i)})$$

is non-stationary in α. In other words the set

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. The proof of Theorem 1 is complete.

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